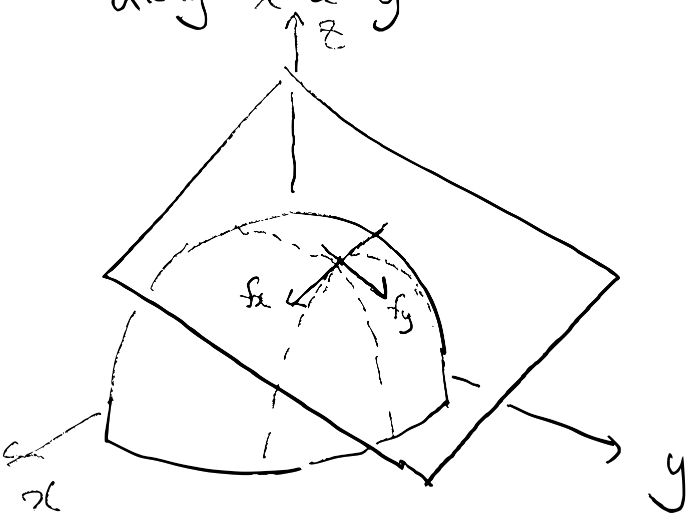
Directional derivatives

Partial derivatives represent tangent lines along x & y-directions.



Directional derivatives: partial derivative along any direction.

<u>Def</u> Let $\vec{u} = \langle a, b \rangle$ be a vector.
The <u>directional derivative</u> of f in the direction of $\vec{u}$ is

$$D_{\vec{u}} f = a f_x + b f_y = \vec{u} \cdot \langle f_x, f_y \rangle.$$

The vector $\langle f_x, f_y \rangle$ is called the <u>gradient</u> of f, denoted ∇f.

Example $f(x,y) = x\sqrt{y}$, $\vec{u} = \langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle$

The directional derivative of f at (2,4) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(2,4) = \vec{u} \cdot \nabla f(2,4)$$

$$= \langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle \cdot \langle f_x(2,4), f_y(2,4) \rangle$$

Note $f_x(x,y) = \sqrt{y} \rightsquigarrow f_x(2,4) = 2$

$f_y(x,y) = \frac{x}{2\sqrt{y}} \rightsquigarrow f_y(2,4) = \frac{1}{2}$

$\Rightarrow D_{\vec{u}} f(2,4) = \langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle \cdot \langle 2, \frac{1}{2} \rangle = \frac{4}{\sqrt{5}} - \frac{1}{2\sqrt{5}} = \frac{7}{2\sqrt{5}}$.

When $\vec{u}$ is a unit vector,

$D_{\vec{u}} f$ is the rate of change of $f(x,y)$ as you move toward the direction of $\vec{u}$.

Namely, when $\vec{u}$ is a unit vector,

* $D_{\vec{u}} f$ is the largest when $\vec{u}$ is in the same direction as ∇f: namely, $\vec{u} = \frac{\nabla f}{|\nabla f|}$. Then, $\boxed{D_{\vec{u}} f = |\nabla f|}$.
* $D_{\vec{u}} f$ is the smallest when $\vec{u}$ is in the opposite direction to ∇f: namely, $\vec{u} = -\frac{\nabla f}{|\nabla f|}$. Then, $\boxed{D_{\vec{u}} f = -|\nabla f|}$

Also, $\nabla f(x,y)$ is orthogonal to the tangent line to the level curve at (x,y).

<u>Example</u> Let $f(x,y) = \sqrt{x^2+y^2}$.

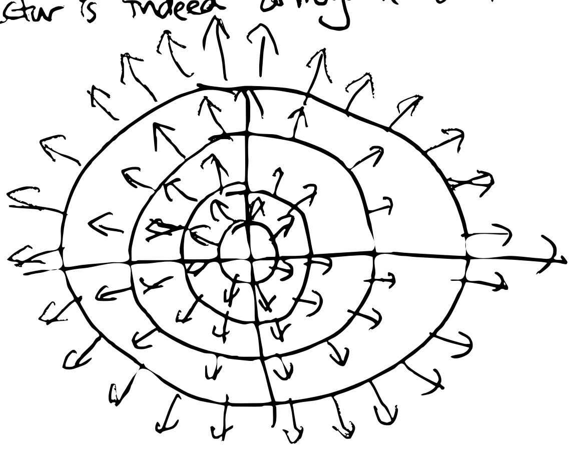
Level curves at level k: $\sqrt{x^2+y^2} = k$, or $x^2+y^2 = k^2$

The gradient vector at (x,y) is

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle.$$

This vector is indeed orthogonal to the tangent line at (x,y).



Three-variables version: $f(x,y,z)$

Gradient: $\boxed{\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle}$

Directional derivative (to the direction of $\vec{u}$)

$$\boxed{D_{\vec{u}} f = \vec{u} \cdot \nabla f}$$

Again, when $\vec{u}$ is a unit vector, $D_{\vec{u}} f$ is the rate of change of $f(x,y,z)$ when you move towards the direction of $\vec{u}$.

* $D_{\vec{u}} f(x,y,z)$ is maximized when $\vec{u}$ is in the same direction as $\nabla f(x,y,z)$. Namely, when $\vec{u} = \frac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|}$.

  In this case, $D_{\vec{u}} f(x,y,z) = |\nabla f(x,y,z)|$

* $D_{\vec{u}} f(x,y,z)$ is minimized when $\vec{u}$ is in the opposite direction to $\nabla f(x,y,z)$. Namely, when $\vec{u} = -\frac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|}$.

  In this case, $D_{\vec{u}} f(x,y,z) = -|\nabla f(x,y,z)|$

* A level surface of $f(x,y,z)$ of level k is the surface $f(x,y,z) = k$. If (a,b,c) is on the level surface, then $\nabla f(a,b,c)$ is orthogonal to the tangent plane of the level surface $\{f(x,y,z) = k\}$ at (a,b,c)

Example Let $f(x,y,z) = 2x^4y^2 - xyz^2 + x^2yz$

$D_{\vec{u}} f(1,1,1)$ for $\vec{u} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$?

$f_x(x,y,z) = 8x^3y^2 - yz^2 + 2xyz$

$\rightsquigarrow f_x(1,1,1) = 8 - 1 + 2 = 9$

$f_y(x,y,z) = 4x^4y - xz^2 + x^2z$

$\rightsquigarrow f_y(1,1,1) = 4 - 1 + 1 = 4$

$f_z(x,y,z) = -2xyz + x^2y$

$\rightsquigarrow f_z(1,1,1) = -2 + 1 = -1$

$\Rightarrow \nabla f(1,1,1) = \langle 9, 4, -1 \rangle$

$\Rightarrow D_{\vec{u}} f(1,1,1) = \vec{u} \cdot \nabla f(1,1,1) = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle \cdot \langle 9, 4, -1 \rangle$

$= \frac{2}{3} \times 9 - \frac{2}{3} \times 4 + \frac{1}{3} \times (-1)$

$= \frac{9}{3} = 3.$

Example The level surface to $f(x,y,z)=\sqrt{x^2+y^2+z^2}$

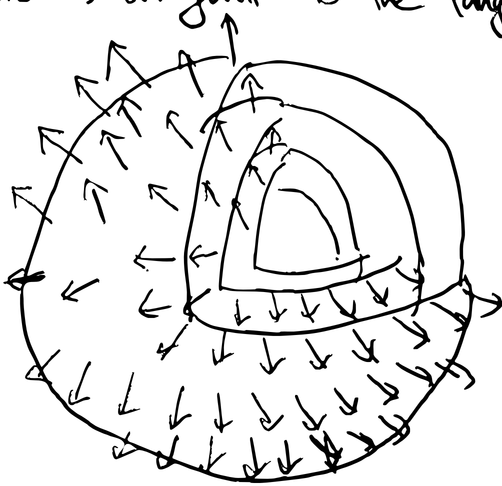
are surfaces of the form $\sqrt{x^2+y^2+z^2}=k$, so $x^2+y^2+z^2=k^2$

are spheres.

$$\nabla f(x,y,z)=\langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$

$$=\left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

which is orthogonal to the tangent plane at (x,y,z).



In other words, the tangent plane to the level surface

$f(x,y,z)=k$ at (a,b,c) has equation

$$f_x(a,b,c)(x-a)+f_y(a,b,c)(y-b)+f_z(a,b,c)(z-c)=0$$

Intuition: ∇f gives the direction to which f increases most rapidly.

For a contour map of a mountain: the steepest uphill.